

TURÁN TYPE INEQUALITIES FOR TRICOMI CONFLUENT HYPERGEOMETRIC FUNCTIONS

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ABSTRACT. Some sharp two-sided Turán type inequalities for parabolic cylinder functions and Tricomi confluent hypergeometric functions are deduced. The proofs are based on integral representations for quotients of parabolic cylinder functions and Tricomi confluent hypergeometric functions, which arise in the study of the infinite divisibility of the Fisher-Snedecor F distribution. Moreover, some complete monotonicity results are given concerning Turán determinants of Tricomi confluent hypergeometric functions. These complement and improve some of the results of Ismail and Laforgia [23].

1. INTRODUCTION

Since the publication in 1948 by Szegő [37] of Turán's inequality for Legendre polynomials [39], many researchers have produced analogous results for orthogonal polynomials and special functions. In the last six decades it was shown by several researchers that the most important special functions satisfy Turán type inequalities, see for example the most recent papers on this topic written in the last five years [3]–[16], [23, 27, 28, 34] and the references therein. Turán type inequalities seem to be evergreen in the theory of special functions, nowadays they have an extensive literature and some of the results have been applied successfully in problems which arise in information theory, economic theory and biophysics. For more details the interested reader is referred to the papers [10, 15, 33, 36]. Motivated by these applications, recently the Turán type inequalities have been investigated also for hypergeometric and confluent hypergeometric functions, as well as for the generalized hypergeometric functions. See [6, 9, 15, 28] and the references therein for more details.

In this paper we make a contribution to the above mentioned results by proving the corresponding sharp Turán type inequalities for parabolic cylinder functions and Tricomi confluent hypergeometric functions. These results naturally complement the earlier results for Hermite polynomials, modified Bessel functions of the second kind and Kummer confluent hypergeometric functions.

In Section 2 we consider the parabolic cylinder functions and we prove a sharp Turán type inequality by using an integral formula from [22]. In Section 3 we establish Turán type inequalities for the Tricomi ψ (confluent hypergeometric) functions by using another integral representation formula from [22]. The latter integral representation was the main tool in the proof of the infinite divisibility of the Fisher-Snedecor F distribution. Finally, in Section 4 we present a general result concerning Turán determinants whose entries are functions having convenient integral representation. These yield complete monotonicity results for Turán determinants of Tricomi

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confluent hypergeometric functions. The main results of Sections 3 and 4 complement and improve the results of Ismail and Laforgia [23] concerning the Tricomi hypergeometric function.

2. TURÁN TYPE INEQUALITIES FOR PARABOLIC CYLINDER FUNCTIONS

The parabolic cylinder function or sometimes called as Weber function $U(a, \cdot)$, denoted also as $D_{-a-1/2}$ following Whittaker's notation, is a particular solution of Weber's differential equation (see [1, p. 687] or [18, p. 116])

$$(2.1) \quad w''(x) - \left(a + \frac{x^2}{4}\right) w(x) = 0$$

and its value is represented explicitly as

$$U(a, x) = \frac{1}{2\eta\sqrt{\pi}} \left[\cos(\eta\pi) \Gamma\left(\frac{1}{2} - \eta\right) y_1(a, x) - \sqrt{2} \sin(\eta\pi) \Gamma(1 - \eta) y_2(a, x) \right],$$

where

$$y_1(a, x) = \exp\left(-\frac{x^2}{4}\right) \Phi\left(\frac{a}{2} + \frac{1}{4}, \frac{1}{2}, \frac{x^2}{2}\right)$$

and

$$y_2(a, x) = x \exp\left(-\frac{x^2}{4}\right) \Phi\left(\frac{a}{2} + \frac{3}{4}, \frac{3}{2}, \frac{x^2}{2}\right)$$

are independent solutions of (2.1), $\eta = a/2 + 1/4$ and $\Phi(a, c, \cdot)$ stands for the Kummer confluent hypergeometric function, called also as confluent hypergeometric function of the first kind.

Our first main result is Theorem 1 below.

Theorem 1. *If $a > 0$ and $x \in \mathbb{R}$, then the following Turán type inequalities are valid*

$$(2.2) \quad 0 < D_{-a}^2(x) - D_{-a-1}(x)D_{-a+1}(x) \leq \mu_a,$$

where

$$\mu_a = \frac{\pi}{2a} \left[\frac{1}{\Gamma^2\left(\frac{a+1}{2}\right)} - \frac{1}{\Gamma\left(\frac{a}{2}\right)\Gamma\left(\frac{a}{2} + 1\right)} \right].$$

The left-hand side of (2.2) is sharp as $|x| \rightarrow \infty$ and it is also valid when $a = 0$ and $x > 0$. The equality is attained on the right-hand side of (2.2) when $x = 0$.

Proof. Let us consider the Turánian

$$\begin{aligned} {}_D\Delta_a(x) &:= D_{-a}^2(x) - D_{-a-1}(x)D_{-a+1}(x) \\ &= U^2\left(a - \frac{1}{2}, x\right) - U\left(a - \frac{3}{2}, x\right)U\left(a + \frac{1}{2}, x\right), \end{aligned}$$

which in view of the differential recurrence relations [1, p. 688]

$$U'(a, x) + \frac{x}{2}U(a, x) + \left(a + \frac{1}{2}\right)U(a + 1, x) = 0$$

and

$$U'(a, x) - \frac{x}{2}U(a, x) + U(a - 1, x) = 0$$

can be rewritten as

$${}_D\Delta_a(x) = \left(1 + \frac{x^2}{4a}\right) U^2\left(a - \frac{1}{2}, x\right) - \frac{1}{a} \left[U'\left(a - \frac{1}{2}, x\right) \right]^2.$$

On the other hand, since $U(a, x)$ satisfies the Weber differential equation (2.1), we obtain

$$U''\left(a - \frac{1}{2}, x\right) = \left(a - \frac{1}{2} + \frac{x^2}{4}\right) U\left(a - \frac{1}{2}, x\right).$$

Moreover, the following integral representation formula [22, p. 885] is valid

$$(2.3) \quad \frac{D_{-a-1}(\sqrt{z})}{\sqrt{z}D_{-a}(\sqrt{z})} = \frac{1}{\sqrt{2\pi}\Gamma(a+1)} \int_0^\infty \frac{|D_{-a}(i\sqrt{t})|^{-2}}{(z+t)\sqrt{t}} dt,$$

where $a > 0$ and $|\arg z| < \pi$. Now, by using (2.3) we get

$$\begin{aligned} {}_D\Delta'_a(x) &= \frac{1}{a}U^2\left(a - \frac{1}{2}, x\right) \left[\frac{x}{2} + \frac{U'(a - \frac{1}{2}, x)}{U(a - \frac{1}{2}, x)} \right] \\ &= -U^2\left(a - \frac{1}{2}, x\right) \left[\frac{U(a + \frac{1}{2}, x)}{U(a - \frac{1}{2}, x)} \right] \\ &= -D_{-a}^2(x) \left[\frac{D_{-a-1}(x)}{D_{-a}(x)} \right] \\ &= -D_{-a}^2(x) \int_0^\infty \frac{x}{x^2+t} \varphi_a(t) dt, \end{aligned}$$

where

$$\varphi_a(t) = \frac{|D_{-a}(i\sqrt{t})|^{-2}}{\sqrt{2\pi}\Gamma(a+1)\sqrt{t}}.$$

Thus, the function $x \mapsto {}_D\Delta_a(x)$ is increasing on $(-\infty, 0]$ and decreasing on $[0, \infty)$, and consequently by using [1, p. 687]

$$U(a, 0) = \frac{\sqrt{\pi}}{2^{\frac{a}{2} + \frac{1}{4}} \Gamma(\frac{a}{2} + \frac{3}{4})}$$

we have ${}_D\Delta_a(x) \leq {}_D\Delta_a(0) = \mu_a$ for all $x \in \mathbb{R}$ and $a > 0$. This proves the inequality on the right-hand side of (2.2). Now, for the inequality on the left-hand side of (2.2) recall that for $|\arg z| < \pi/2$ we have [1, p. 689]

$$\lim_{|z| \rightarrow \infty} \frac{U(a, z)}{e^{-\frac{z^2}{4}} z^{-a-\frac{1}{2}}} = 1,$$

and then $\lim_{|x| \rightarrow \infty} {}_D\Delta_a(x) = 0$, which completes the proof when $a > 0$.

Finally, recall that [1, p. 692]

$$U\left(-\frac{1}{2}, x\right) = e^{-\frac{x^2}{4}}, \quad U\left(-\frac{3}{2}, x\right) = xe^{-\frac{x^2}{4}}$$

and

$$U\left(\frac{1}{2}, x\right) = \sqrt{\frac{\pi}{2}} e^{\frac{x^2}{4}} \operatorname{erfc}\left(\frac{x}{\sqrt{2}}\right),$$

where erfc , defined by

$$\operatorname{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2} dt,$$

denotes the complementary error function. Thus, we obtain for all $x > 0$ that

$${}_D\Delta_0(x) = D_0^2(x) - D_{-1}(x)D_1(x) = e^{-\frac{x^2}{2}} - x \int_x^\infty e^{-\frac{t^2}{2}} dt > 0,$$

which is exactly the upper bound inequality of Gordon [19] for the Mills ratio of the standard normal distribution. For more details see also [8, p. 1363]. This completes the proof. \square

Remark 1. We mention that recently, by using a different approach, Segura [35, Theorem 11] proved the inequalities

$$1 < \frac{D_{-a}^2(x)}{D_{-a-1}(x)D_{-a+1}(x)} < \sqrt{\frac{a+1}{a-1}},$$

where $a > 0$ and $x \in \mathbb{R}$ on the left-hand side, and $a > 1$ and $x \in \mathbb{R}$ on the right-hand side. We also note here that the inequality on the left-hand side of (2.2) complements the Turán type inequality for Hermite polynomials (see [31, 32, 37])

$$H_n^2(x) - H_{n-1}(x)H_{n+1}(x) \geq 0,$$

which is valid for all real x and $n \in \{1, 2, \dots\}$. Indeed, if n is a non-negative integer, then by using the relation [1, p. 780]

$$D_n(x) = 2^{-\frac{n}{2}} e^{-\frac{x^2}{4}} H_n\left(\frac{x}{\sqrt{2}}\right),$$

the above Turán type inequality becomes

$$D_n^2(x) - D_{n-1}(x)D_{n+1}(x) \geq 0,$$

where $n \in \{1, 2, \dots\}$ and $x \in \mathbb{R}$. The left-hand side of (2.2) complements this inequality.

Now, let us consider the cases when $a \in \{-3/2, -1/2\}$. Observe that in view of [1, p. 692]

$$U(0, x) = \sqrt{\frac{x}{2\pi}} K_{\frac{1}{4}}(z), \quad U(-1, x) = \frac{x\sqrt{x}}{2\sqrt{2\pi}} \left[K_{\frac{1}{4}}(z) + K_{\frac{3}{4}}(z) \right],$$

$$U(-2, x) = \frac{x^2\sqrt{x}}{4\sqrt{2\pi}} \left[2K_{\frac{1}{4}}(z) + 3K_{\frac{3}{4}}(z) - K_{\frac{5}{4}}(z) \right]$$

and

$$U(-3, x) = \frac{x^3\sqrt{x}}{8\sqrt{2\pi}} \left[5K_{\frac{1}{4}}(z) + 9K_{\frac{3}{4}}(z) - 5K_{\frac{5}{4}}(z) - K_{\frac{7}{4}}(z) \right]$$

we have that

$$\begin{aligned} {}_D\Delta_{-\frac{3}{2}}(x) &= U^2(-2, x) - U(-3, x)U(-1, x) \\ &= \frac{x^5}{32\pi} \left[K_{\frac{3}{4}}(z) \left(K_{\frac{7}{4}}(z) - K_{\frac{5}{4}}(z) \right) + K_{\frac{5}{4}}^2(z) - K_{\frac{1}{4}}^2(z) \right. \\ &\quad \left. + K_{\frac{1}{4}}(z) \left(K_{\frac{5}{4}}(z) + K_{\frac{7}{4}}(z) - 2K_{\frac{3}{4}}(z) \right) \right], \end{aligned}$$

and

$$\begin{aligned} {}_D\Delta_{-\frac{1}{2}}(x) &= U^2(-1, x) - U(-2, x)U(0, x) \\ &= \frac{x^3}{8\pi} \left[K_{\frac{1}{4}}(z) \left(K_{\frac{5}{4}}(z) - K_{\frac{3}{4}}(z) \right) + K_{\frac{3}{4}}^2(z) - K_{\frac{1}{4}}^2(z) \right], \end{aligned}$$

where $z = x^2/4$ and K_a stands for the modified Bessel function of the second kind. By using the known fact (see [29]) that $a \mapsto K_a(x)$ is increasing on $(0, \infty)$ for each fixed $x > 0$, it follows that ${}_D\Delta_{-\frac{3}{2}}(x) > 0$ and ${}_D\Delta_{-\frac{1}{2}}(x) > 0$ for all $x > 0$. Numerical experiments and the above results for $a \in \{-3/2, -1/2, 0\}$ suggest that the left-hand side of the inequality (2.2) is also valid for all $a \leq 0$ and $x > 0$, however, we were unable to prove this.

3. TURÁN TYPE INEQUALITIES FOR TRICOMI ψ FUNCTION

The Tricomi's confluent hypergeometric function, also called confluent hypergeometric function of the second kind, $\psi(a, c, \cdot)$ is a particular solution of the confluent hypergeometric differential equation (see [1, p. 504] or [17, p. 248])

$$(3.1) \quad xw''(x) + (c - x)w'(x) - aw(x) = 0$$

and its value is defined in terms of the Kummer confluent hypergeometric function as

$$\psi(a, c, x) = \frac{\Gamma(1 - c)}{\Gamma(a - c + 1)} \Phi(a, c, x) + \frac{\Gamma(c - 1)}{\Gamma(a)} x^{1-c} \Phi(a - c + 1, 2 - c, x).$$

Now, recall the following Turán type inequalities, which hold for all $a > 1$ and $x > 0$

$$(3.2) \quad \frac{1}{1-a} K_a^2(x) < K_a^2(x) - K_{a-1}(x) K_{a+1}(x) < 0.$$

Moreover, the right-hand side of (3.2) holds true for all $a \in \mathbb{R}$. These inequalities are sharp in the sense that the constants $1/(1-a)$ and 0 are best possible.

For the sake of completeness it should be mentioned that the right-hand side of (3.2) was first proved independently by Ismail and Muldoon [24] and van Haeringen [40], and rediscovered later by Laforgia and Natalini [30]. Note that in [24] the authors actually proved that for all fixed $x > 0$ and $b > 0$, the function $a \mapsto K_{a+b}(x)/K_a(x)$ is increasing on \mathbb{R} . Another proof of the right-hand side of (3.2), with proper credit, is in [11]. Recently, Baricz [12] and Segura [34], proved the two sided inequality in (3.2) by using different approaches. See also [14] for more details on (3.2). We also note here that the left-hand side of (3.2) provides actually an upper bound for the effective variance of the generalized Gaussian distribution. More precisely, in [2] the authors used (without proof) the inequality $0 < v_{\text{eff}} < 1/(\mu - 1)$ for $\mu = a + 4$, where

$$v_{\text{eff}} = \frac{K_{\mu-1}(x) K_{\mu+1}(x)}{K_{\mu}^2(x)} - 1$$

is the effective variance of the generalized Gaussian distribution.

Observe that by using the relation [1, p. 510]

$$K_a(x) = \sqrt{\pi} (2x)^a e^{-x} \psi\left(a + \frac{1}{2}, 2a + 1, 2x\right)$$

the Turán type inequality (3.2) for $a > 1$ and $x > 0$ can be rewritten as

$$(3.3) \quad \frac{1}{1-a} \psi^2\left(a + \frac{1}{2}, 2a + 1, x\right) < \Delta_a(x) < 0,$$

where

$$\Delta_a(x) := \psi^2\left(a + \frac{1}{2}, 2a + 1, x\right) - \psi\left(a - \frac{1}{2}, 2a - 1, x\right) \psi\left(a + \frac{3}{2}, 2a + 3, x\right).$$

The next result complements the above inequality.

Theorem 2. *If $a > 0 > c$ and $x > 0$, then the following sharp Turán type inequalities are valid*

$$(3.4) \quad \frac{1}{c} \psi^2(a, c, x) < \psi^2(a, c, x) - \psi(a - 1, c - 1, x) \psi(a + 1, c + 1, x) < 0.$$

Moreover, the right-hand side of (3.4) holds true for all $a > 0$, $c < 1$ and $x > 0$. These inequalities are sharp in the sense that the constants $1/c$ and 0 are best possible.

Proof. First consider the expression

$${}_{\psi} \Delta_{a,c}(x) := \psi^2(a, c, x) - \psi(a - 1, c - 1, x) \psi(a + 1, c + 1, x),$$

which in view of the relations [1, p. 507]

$$\psi'(a, c, x) = -a \psi(a + 1, c + 1, x),$$

$$\psi(a - 1, c - 1, x) = (1 - c + x) \psi(a, c, x) - x \psi'(a, c, x)$$

can be rewritten as

$${}_{\psi} \Delta_{a,c}(x) = \psi^2(a, c, x) + \frac{1}{a} (1 - c + x) \psi(a, c, x) \psi'(a, c, x) - \frac{x}{a} [\psi'(a, c, x)]^2.$$

On the other hand, because $\psi(a, c, x)$ satisfies the confluent differential equation (3.1), we obtain

$$(3.5) \quad \left[\frac{x \psi'(a, c, x)}{\psi(a, c, x)} \right]' = (1 + x - c) \frac{\psi'(a, c, x)}{\psi(a, c, x)} + a - x \left[\frac{\psi'(a, c, x)}{\psi(a, c, x)} \right]^2$$

and conclude that

$$\frac{\psi \Delta_{a,c}(x)}{\psi^2(a, c, x)} = \frac{1}{a} \left[\frac{x\psi'(a, c, x)}{\psi(a, c, x)} \right]' = - \left[\frac{x\psi(a+1, c+1, x)}{\psi(a, c, x)} \right]'$$

For $|\arg z| < \pi$, $a > 0$ and $c < 1$ the integral representation [22, p. 885]

$$(3.6) \quad \frac{\psi(a+1, c+1, z)}{\psi(a, c, z)} = \int_0^\infty \frac{t^{-c} e^{-t} |\psi(a, c, te^{i\pi})|^{-2}}{(z+t)\Gamma(a+1)\Gamma(a-c+1)} dt$$

is valid. By using the notation

$$\varphi_{a,c}(t) := \frac{t^{-c} e^{-t} |\psi(a, c, te^{i\pi})|^{-2}}{\Gamma(a+1)\Gamma(a-c+1)},$$

(3.6) implies that

$$\frac{\psi \Delta_{a,c}(x)}{\psi^2(a, c, x)} = - \left[\int_0^\infty \frac{x}{x+t} \varphi_{a,c}(t) dt \right]' = - \int_0^\infty \frac{t \varphi_{a,c}(t) dt}{(x+t)^2} < 0$$

and

$$\left[\frac{\psi \Delta_{a,c}(x)}{\psi^2(a, c, x)} \right]' = - \left[\int_0^\infty \frac{t \varphi_{a,c}(t) dt}{(x+t)^2} \right]' = \int_0^\infty \frac{2t \varphi_{a,c}(t) dt}{(x+t)^3} > 0$$

for all $a > 0$, $c < 1$ and $x > 0$. Thus, the function $x \mapsto \psi \Delta_{a,c}(x)/\psi^2(a, c, x)$ maps $(0, \infty)$ into $(-\infty, 0)$ and it is strictly increasing. Hence, we obtain for all $a > 0$, $c < 1$ and $x > 0$

$$\alpha_{a,c} := \lim_{x \rightarrow 0} \frac{\psi \Delta_{a,c}(x)}{\psi^2(a, c, x)} < \frac{\psi \Delta_{a,c}(x)}{\psi^2(a, c, x)} < \lim_{x \rightarrow \infty} \frac{\psi \Delta_{a,c}(x)}{\psi^2(a, c, x)} =: \beta_{a,c}.$$

The asymptotic expansion [1, p. 508]

$$(3.7) \quad \psi(a, c, x) \sim x^{-a} \left(1 + a(c-a-1) \frac{1}{x} + \frac{1}{2} a(a+1)(a+1-c)(a+2-c) \frac{1}{x^2} + \dots \right),$$

which is valid for large real x and fixed a and c , implies that $\beta_{a,c} = 0$. Similarly, by using the asymptotic expansion [1, p. 508]

$$(3.8) \quad \psi(a, c, x) \sim \frac{\Gamma(1-c)}{\Gamma(1+a-c)},$$

where $c < 1$ and $a > 0$ are fixed and $x \rightarrow 0$, we see that $\alpha_{a,c} = 1/c$ for $c < 0$. It is clear by construction that the constants $\alpha_{a,c}$ and $\beta_{a,c}$ are best possible. \square

Remark 2. Observe that by using [1, p. 505]

$$(3.9) \quad W_{\kappa,\mu}(x) = \exp\left(-\frac{x}{2}\right) x^{\mu+\frac{1}{2}} \psi\left(\mu - \kappa + \frac{1}{2}, 1 + 2\mu, x\right),$$

for $x > 0$ the inequality (3.4) can be rewritten in terms of Whittaker functions $W_{\kappa,\mu}$ as follows

$$(3.10) \quad \frac{1}{1+2\mu} W_{\kappa,\mu}^2(x) < W_{\kappa,\mu}^2(x) - W_{\kappa+\frac{1}{2},\mu-\frac{1}{2}}(x) W_{\kappa-\frac{1}{2},\mu+\frac{1}{2}}(x) < 0,$$

where the left-hand side holds for $0 > \mu + 1/2 > \kappa$, while the right-hand side is valid for $1/2 > \mu + 1/2 > \kappa$.

Remark 3. We note that by using (3.6) directly we can prove a weaker Turán type inequality than the right-hand side of (3.4). More precisely, because of (3.6) (see also [22, p. 889]) the function

$$x \mapsto G(x) := -\frac{\psi'(a, c, x)}{\psi(a, c, x)} = \frac{a\psi(a+1, c+1, x)}{\psi(a, c, x)}$$

is a Stieltjes transform, and consequently it is strictly completely monotonic, i.e. for all $a > 0$, $c < 1$ and $x > 0$ we have $(-1)^n G^{(n)}(x) > 0$, which in particular implies that the function $x \mapsto \psi'(a, c, x)/\psi(a, c, x)$ is increasing and then the Laguerre type inequality

$$\psi''(a, c, x)\psi(a, c, x) - [\psi'(a, c, x)]^2 < 0$$

is valid. This is equivalent to

$$a\psi^2(a+1, c+1, x) - (a+1)\psi(a, c, x)\psi(a+2, c+2, x) < 0$$

or to

$$(3.11) \quad \psi^2(a, c, x) - \psi(a-1, c-1, x)\psi(a+1, c+1, x) < \frac{1}{a}\psi^2(a, c, x),$$

where $a > 1$, $c < 2$ and $x > 0$.

Moreover, by using [1, p. 505]

$$(3.12) \quad \psi(a, c, x) = \frac{1}{\Gamma(a)} \int_0^\infty e^{-xt} t^{a-1} (1+t)^{c-a-1} dt,$$

the restriction $c < 2$ in the inequality (3.11) can be removed. More precisely, the Hölder-Rogers inequality for integrals implies for all $a_1, a_2 > 0$, $c_1, c_2 \in \mathbb{R}$, $x > 0$ and $\alpha \in [0, 1]$

$$\begin{aligned} & \Gamma(\alpha a_1 + (1-\alpha)a_2)\psi(\alpha a_1 + (1-\alpha)a_2, \alpha c_1 + (1-\alpha)c_2, x) \\ &= \int_0^\infty e^{-xt} t^{\alpha a_1 + (1-\alpha)a_2 - 1} (1+t)^{\alpha c_1 + (1-\alpha)c_2 - (\alpha a_1 + (1-\alpha)a_2) - 1} dt \\ &= \int_0^\infty (e^{-xt} t^{a_1 - 1} (1+t)^{c_1 - a_1 - 1})^\alpha (e^{-xt} t^{a_2 - 1} (1+t)^{c_2 - a_2 - 1})^{1-\alpha} dt \\ &< \left[\int_0^\infty e^{-xt} t^{a_1 - 1} (1+t)^{c_1 - a_1 - 1} dt \right]^\alpha \left[\int_0^\infty e^{-xt} t^{a_2 - 1} (1+t)^{c_2 - a_2 - 1} dt \right]^{1-\alpha} \\ &= [\Gamma(a_1)\psi(a_1, c_1, x)]^\alpha [\Gamma(a_2)\psi(a_2, c_2, x)]^{1-\alpha} \end{aligned}$$

and then the two-variable function $(a, c) \mapsto \Gamma(a)\psi(a, c, x)$ is strictly logarithmically convex for each $a, x > 0$ and $c \in \mathbb{R}$. Now, observe that the above inequality in particular for $\alpha = 1/2$, $a_1 = a-1$, $a_2 = a+1$, $c_1 = c-1$ and $c_2 = c+1$ reduces to (3.11).

Remark 4. Observe that by using the above idea mutatis mutandis the function $a \mapsto \Gamma(a)\psi(a, c, x)$ is also strictly logarithmically convex for $a, x > 0$ and $c \in \mathbb{R}$, and consequently the Turán type inequality

$$(3.13) \quad \psi^2(a, c, x) - \psi(a-1, c, x)\psi(a+1, c, x) < \frac{1}{a}\psi^2(a, c, x)$$

is valid for all $a > 1$ and $c, x \in \mathbb{R}$. Taking into account the relation [1, p. 510]

$$D_{-a}(x) = 2^{-\frac{a}{2}} \exp\left(-\frac{x^2}{4}\right) \psi\left(\frac{a}{2}, \frac{1}{2}, \frac{x^2}{2}\right)$$

the above inequality in particular reduces to

$$(3.14) \quad D_{-2a}^2(x) - D_{-2a-2}(x)D_{-2a+2}(x) < \frac{1}{a}D_{-2a}^2(x),$$

which resembles to (2.2). However, in the above Turán type inequalities the constant $1/a$ is not best possible, as we shall see below.

The next result is similar to (3.4).

Theorem 3. *If $a > 1 > c$ and $x > 0$, then the next sharp Turán type inequality is valid*

$$(3.15) \quad \frac{1}{1+a-c}\psi^2(a, c, x) > \psi^2(a, c, x) - \psi(a-1, c, x)\psi(a+1, c, x) > 0.$$

Moreover, the right-hand side of (3.15) holds true for all $a > 0$, $c < 1$ and $x > 0$. These inequalities are sharp in the sense that the constants $1/(1+a-c)$ and 0 are best possible.

Proof. The proof is very similar to the proof of (3.4), so we only sketch the proof. By using the recurrence relations [1, p. 507]

$$\psi(a-1, c, x) = (a-c+x)\psi(a, c, x) - x\psi'(a, c, x)$$

and

$$a(1+a-c)\psi(a+1, c, x) = a\psi(a, c, x) + x\psi'(a, c, x),$$

the expression

$${}_{\psi}\Delta_a(x) := \psi^2(a, c, x) - \psi(a-1, c, x)\psi(a+1, c, x)$$

can be rewritten as

$${}_{\psi}\Delta_a(x) = \frac{1-x}{1+a-c}\psi^2(a, c, x) - \frac{x(x-c)}{a(1+a-c)}\psi(a, c, x)\psi'(a, c, x) + \frac{x^2}{a(1+a-c)}[\psi'(a, c, x)]^2.$$

In view of (3.5) and (3.6) this implies that

$$\frac{(1+a-c){}_{\psi}\Delta_a(x)}{\psi^2(a, c, x)} = 1 + \frac{x\psi'(a, c, x)}{a\psi(a, c, x)} - \frac{x}{a} \left[\frac{x\psi'(a, c, x)}{\psi(a, c, x)} \right]' = 1 - \int_0^\infty \frac{x^2\varphi_{a,c}(t)dt}{(x+t)^2}$$

and then

$$\left[\frac{(1+a-c){}_{\psi}\Delta_a(x)}{\psi^2(a, c, x)} \right]' = - \int_0^\infty \frac{2xt\varphi_{a,c}(t)dt}{(x+t)^3} < 0$$

for all $a > 0$, $c < 1$ and $x > 0$. Consequently the function $x \mapsto {}_{\psi}\Delta_a(x)/\psi^2(a, c, x)$ is strictly decreasing on $(0, \infty)$, which implies that for all $a > 0$, $c < 1$ and $x > 0$ we have

$$\alpha_a := \lim_{x \rightarrow 0} \frac{{}_{\psi}\Delta_a(x)}{\psi^2(a, c, x)} > \frac{{}_{\psi}\Delta_a(x)}{\psi^2(a, c, x)} > \lim_{x \rightarrow \infty} \frac{{}_{\psi}\Delta_a(x)}{\psi^2(a, c, x)} =: \beta_a,$$

where $\alpha_a = 1/(1+a-c)$ and $\beta_a = 0$ in view of the asymptotic expansions (3.7) and (3.8). Moreover, the right-hand side of the above inequality is valid for all $a > 0$, $c < 1$ and $x > 0$. \square

Remark 5. Observe that the left-hand side of the Turán type inequality (3.15) improves the inequality (3.13), and the constant $1/(1+a-c)$ cannot be improved. Moreover, we note that in particular the inequality (3.15) becomes

$$0 < D_{-2a}^2(x) - D_{-2a-2}(x)D_{-2a+2}(x) < \frac{1}{a+1/2}D_{-2a}^2(x),$$

where $a > 0$ and $x > 0$ on the left-hand side, and $a > 1$ and $x > 0$ on the right-hand side. Clearly, the right-hand side of this inequality is an improvement over (3.14) and the constant $1/(a+1/2)$ is optimum.

Finally, observe that by using (3.9) for $x > 0$ the inequality (3.15) can be rewritten as follows

$$\frac{1}{-\mu - \kappa + 1/2}W_{\kappa, \mu}^2(x) > W_{\kappa, \mu}^2(x) - W_{\kappa-1, \mu}(x)W_{\kappa+1, \mu}(x) > 0,$$

where the left-hand side is valid for $-1/2 > \mu - 1/2 > \kappa$, while the right-hand side is valid for $1/2 > \mu + 1/2 > \kappa$.

The following result is a companion of (3.4) and (3.15).

Theorem 4. *The function $c \mapsto \psi(a, c, x)$ is strictly logarithmically convex on \mathbb{R} for all $a, x > 0$ fixed, and the following sharp Turán type inequality*

$$(3.16) \quad \frac{a}{c(1+a-c)}\psi^2(a, c, x) < \psi^2(a, c, x) - \psi(a, c-1, x)\psi(a, c+1, x) < 0$$

is valid for all $a > 0 > c$ and $x > 0$. Furthermore, the right-hand side of (3.16) holds for all $a, x > 0$ and $c \in \mathbb{R}$. These inequalities are sharp in the sense that the constants $a(c(1+a-c))^{-1}$ and 0 are best possible. In addition, the sharp inequality

$$(3.17) \quad \frac{1}{2-c}\psi^2(a, c, x) < \psi^2(a, c, x) - \psi(a, c-1, x)\psi(a, c+1, x) < 0$$

is also valid for all $x > 0$ and $a > c - 1 > 1$ in the case of the left-hand side, and $a > c - 1 > 0$ in the case of the right-hand side. These inequalities are sharp in the sense that the constants $1/(2 - c)$ and 0 are best possible.

Proof. The proof of the strict logarithmic convexity of $c \mapsto \psi(a, c, x)$ goes along the lines outlined in Remark 3, so we shall omit the details. The right-hand side of (3.16) follows from this strict logarithmic convexity property, however, we give here an alternative proof, which is similar to the proof of (3.4). For this consider the Turánian

$$\psi \Delta_c(x) := \psi^2(a, c, x) - \psi(a, c - 1, x)\psi(a, c + 1, x),$$

which by using the relations [1, p. 507]

$$(1 + a - c)\psi(a, c - 1, x) = (1 - c)\psi(a, c, x) - x\psi'(a, c, x)$$

and

$$\psi(a, c + 1, x) = \psi(a, c, x) - \psi'(a, c, x),$$

can be rewritten as

$$\psi \Delta_c(x) = \frac{a}{1 + a - c} \psi^2(a, c, x) + \frac{1 + x - c}{1 + a - c} \psi(a, c, x) \psi'(a, c, x) - \frac{x}{1 + a - c} [\psi'(a, c, x)]^2.$$

Consequently we have

$$\frac{(1 + a - c)\psi \Delta_c(x)}{\psi^2(a, c, x)} = a + (1 + x - c) \frac{\psi'(a, c, x)}{\psi(a, c, x)} - x \left[\frac{\psi'(a, c, x)}{\psi(a, c, x)} \right]^2 = \left[\frac{x\psi'(a, c, x)}{\psi(a, c, x)} \right]'$$

which in view of (3.6) implies that

$$\frac{\psi \Delta_c(x)}{\psi^2(a, c, x)} = -\frac{a}{1 + a - c} \int_0^\infty \frac{t\varphi_{a,c}(t)dt}{(x + t)^2} < 0$$

and

$$\left[\frac{\psi \Delta_c(x)}{\psi^2(a, c, x)} \right]' = \frac{a}{1 + a - c} \int_0^\infty \frac{2t\varphi_{a,c}(t)dt}{(x + t)^3} > 0$$

for all $a > 0$, $c < 1$ and $x > 0$. Hence, the function $x \mapsto \psi \Delta_c(x)/\psi^2(a, c, x)$ maps $(0, \infty)$ into $(-\infty, 0)$ and it is strictly increasing. From this we obtain for all $a > 0$, $c < 1$ and $x > 0$

$$\alpha_c := \lim_{x \rightarrow 0} \frac{\psi \Delta_c(x)}{\psi^2(a, c, x)} < \frac{\psi \Delta_c(x)}{\psi^2(a, c, x)} < \lim_{x \rightarrow \infty} \frac{\psi \Delta_c(x)}{\psi^2(a, c, x)} =: \beta_c,$$

where $(1 + a - c)\alpha_c = a/c$ and $\beta_c = 0$ in view of the asymptotic expansions (3.7) and (3.8).

Now, let us focus on the inequality (3.17). By using the Kummer transformation [1, p. 505]

$$(3.18) \quad \psi(a, c, x) = x^{1-c} \psi(1 + a - c, 2 - c, x)$$

the Turán type inequality (3.4) becomes

$$\frac{1}{c} \psi^2(1 + a - c, 2 - c, x) < \psi^2(1 + a - c, 2 - c, x) - \psi(1 + a - c, 3 - c, x) \psi(1 + a - c, 1 - c, x) < 0.$$

Replacing a by $a + c - 1$ we obtain

$$\frac{1}{c} \psi^2(a, 2 - c, x) < \psi^2(a, 2 - c, x) - \psi(a, 3 - c, x) \psi(a, 1 - c, x) < 0.$$

The replacement of c by $2 - c$ gives (3.17). The sharpness of inequality (3.17) follows from the large x asymptotic expansion (3.7) and from the expansion

$$\psi(a, c, x) \sim \frac{\Gamma(c - 1)}{\Gamma(a)} x^{1-c}, \quad \text{as } x \rightarrow 0,$$

which is valid for fixed a and c if $c > 1$. □

Remark 6. We note that in view of (3.9) the Turán type inequality (3.16) for $x > 0$ in terms of Whittaker functions $W_{\kappa,\mu}$ reads as follows

$$-\frac{\mu - \kappa + 1/2}{(1 + 2\mu)(\mu + \kappa + 1/2)} W_{\kappa,\mu}^2(x) < W_{\kappa,\mu}^2(x) - W_{\kappa-\frac{1}{2},\mu-\frac{1}{2}}(x) W_{\kappa+\frac{1}{2},\mu+\frac{1}{2}}(x) < 0,$$

where the left-hand side holds for $0 > \mu + 1/2 > \kappa$, while the right-hand side is valid for $\mu + 1/2 > \kappa$. Similarly, the inequality (3.17) can be rewritten as

$$\frac{1}{1 - 2\mu} W_{\kappa,\mu}^2(x) < W_{\kappa,\mu}^2(x) - W_{\kappa-\frac{1}{2},\mu-\frac{1}{2}}(x) W_{\kappa+\frac{1}{2},\mu+\frac{1}{2}}(x) < 0,$$

where the left-hand side holds for $1 < \mu + 1/2 < 1 - \kappa$, while the right-hand side is valid for $1/2 < \mu + 1/2 < -\kappa$. Moreover, it should be mentioned here that by using the Hölder-Rogers inequality as in Remark 3, the right-hand side of the above inequalities can be generalized in the following way: the two-variable function

$$(\kappa, \mu) \mapsto \Gamma\left(\mu - \kappa + \frac{1}{2}\right) W_{\kappa,\mu}(x) = \exp\left(-\frac{x}{2}\right) x^{\mu+\frac{1}{2}} \int_0^\infty e^{-xt} t^{\mu-\kappa-\frac{1}{2}} (1+t)^{\mu+\kappa-\frac{1}{2}} dt$$

is logarithmically convex for $\mu + 1/2 > \kappa$ and fixed $x \in \mathbb{R}$. Finally, observe that the above strict logarithmic convexity property implies also the inequality

$$W_{\kappa,\mu}^2(x) - W_{\kappa+\frac{1}{2},\mu-\frac{1}{2}}(x) W_{\kappa-\frac{1}{2},\mu+\frac{1}{2}}(x) < \frac{1}{\mu - \kappa + 1/2} W_{\kappa,\mu}^2(x),$$

however, this is weaker than the right-hand side of (3.10).

Remark 7. Observe that the Kummer transformation (3.18) is also useful to prove the right-hand side of (3.4). More precisely, from (3.18) we obtain

$$(3.19) \quad \Gamma(1+a-c)\psi(a, c, x) = x^{1-c}\Gamma(1+a-c)\psi(1+a-c, 2-c, x) = \int_0^\infty x^{1-c} e^{-xt} t^{a-c} (1+t)^{-a} dt$$

and by using the Hölder-Rogers inequality, as in Remark 3, we conclude that the two-variable function $(a, c) \mapsto \Gamma(1+a-c)\psi(a, c, x)$ is strictly logarithmically convex and consequently the right-hand side of the Turán type inequality (3.4) is valid for all $c < a + 1$ and $x > 0$.

4. TURÁN DETERMINANTS OF TRICOMI CONFLUENT HYPERGEOMETRIC FUNCTIONS

In this section we discuss the connection between the present paper and [23]. For this let us consider the determinants

$${}_1\text{Det}_n(x) := \begin{vmatrix} g(a, c, x) & g(a+1, c, x) & \cdots & g(a+n, c, x) \\ g(a+1, c, x) & g(a+2, c, x) & \cdots & g(a+n+1, c, x) \\ \vdots & \vdots & & \vdots \\ g(a+n, c, x) & g(a+n+1, c, x) & \cdots & g(a+2n, c, x) \end{vmatrix}$$

and

$${}_2\text{Det}_n(x) := \begin{vmatrix} g(a, c, x) & g(a, c+1, x) & \cdots & g(a, c+n, x) \\ g(a, c+1, x) & g(a, c+2, x) & \cdots & g(a, c+n+1, x) \\ \vdots & \vdots & & \vdots \\ g(a, c+n, x) & g(a, c+n+1, x) & \cdots & g(a, c+2n, x) \end{vmatrix},$$

where $g(a, c, x) := \Gamma(a)\psi(a, c, x)$. In [23] Ismail and Laforgia stated that for all $a > 0$, $c \in \mathbb{R}$ and $n \in \{0, 1, \dots\}$ the determinants ${}_1\text{Det}_n(x)$ and ${}_2\text{Det}_n(x)$ are completely monotonic on $(0, \infty)$ with respect to x , that is, we have

$$(4.1) \quad (-1)^m {}_1\text{Det}_n^{(m)}(x) \geq 0$$

and

$$(4.2) \quad (-1)^m {}_2\text{Det}_n^{(m)}(x) \geq 0$$

for all $a, x > 0$, $c \in \mathbb{R}$ and $n, m \in \{0, 1, \dots\}$. Observe that if we choose in (4.1) the values $m = 0$, $n = 1$ and instead of a we write $a - 1$, then we obtain the weak Turán type inequality (3.13). Similarly, if we take in (4.2) the values $m = 0$, $n = 1$ and we write $c - 1$ instead of c , then we get the right-hand side of the inequality (3.16).

We now present a general result which is in the same spirit as [23, Remark 2.9] and which generalizes the above mentioned results from [23]. For this consider $\alpha, \beta \in \mathbb{R}$ such that $\alpha < \beta$, and let $\{f_n\}_{n \geq 0}$ be a sequence of functions, defined by

$$(4.3) \quad f_n(x) := \int_{\alpha}^{\beta} [\phi(t, x)]^n d\mu(t, x),$$

where $\phi, \mu : [\alpha, \beta] \times \mathbb{R} \rightarrow \mathbb{R}$. Consider also the determinant

$$\text{Det}_n(x) := \begin{vmatrix} f_0(x) & f_1(x) & \cdots & f_n(x) \\ f_1(x) & f_2(x) & \cdots & f_{n+1}(x) \\ \vdots & \vdots & & \vdots \\ f_n(x) & f_{n+1}(x) & \cdots & f_{2n}(x) \end{vmatrix}.$$

Then the following result is valid.

Theorem 5. *We have the following representation*

$$(4.4) \quad \text{Det}_n(x) = \frac{1}{(n+1)!} \int_{[\alpha, \beta]^{n+1}} \prod_{0 \leq j < k \leq n} [\phi(t_j, x) - \phi(t_k, x)]^2 \prod_{j=0}^n d\mu(t_j, x).$$

Proof. Observe that

$$\begin{aligned} \text{Det}_n(x) &= \int_{[\alpha, \beta]^{n+1}} \begin{vmatrix} 1 & \phi(t_0, x) & \cdots & [\phi(t_0, x)]^n \\ \phi(t_1, x) & [\phi(t_1, x)]^2 & \cdots & [\phi(t_1, x)]^{n+1} \\ \vdots & \vdots & & \vdots \\ [\phi(t_n, x)]^n & [\phi(t_n, x)]^{n+1} & \cdots & [\phi(t_n, x)]^{2n} \end{vmatrix} \prod_{j=0}^n d\mu(t_j, x) \\ &= \int_{[\alpha, \beta]^{n+1}} \begin{vmatrix} 1 & \phi(t_0, x) & \cdots & [\phi(t_0, x)]^n \\ 1 & \phi(t_1, x) & \cdots & [\phi(t_1, x)]^n \\ \vdots & \vdots & & \vdots \\ 1 & \phi(t_n, x) & \cdots & [\phi(t_n, x)]^n \end{vmatrix} \prod_{j=0}^n [\phi(t_j, x)]^j d\mu(t_j, x) \\ &= \text{sign}(\sigma) \int_{[\alpha, \beta]^{n+1}} \begin{vmatrix} 1 & \phi(t_{\sigma(0)}, x) & \cdots & [\phi(t_{\sigma(0)}, x)]^n \\ 1 & \phi(t_{\sigma(1)}, x) & \cdots & [\phi(t_{\sigma(1)}, x)]^n \\ \vdots & \vdots & & \vdots \\ 1 & \phi(t_{\sigma(n)}, x) & \cdots & [\phi(t_{\sigma(n)}, x)]^n \end{vmatrix} \prod_{j=0}^n [\phi(t_j, x)]^{\sigma(j)} d\mu(t_j, x), \end{aligned}$$

where σ is a permutation on $\{0, 1, \dots, n\}$. The determinant in the last expression is a Vandermonde determinant which can be evaluated as a product. Thus, if we add over all possible σ and divide by $(n+1)!$, then we can see that $\text{Det}_n(x)$ is given by the right-hand side of (4.4) because

$$\sum_{\sigma} \text{sign}(\sigma) \prod_{j=0}^n [\phi(t_j, x)]^{\sigma(j)} = \prod_{0 \leq j < k \leq n} [\phi(t_j, x) - \phi(t_k, x)].$$

□

Note that the proof above is similar to Heine's classical proof of his integral representation, see for example [38] and [20]. We also note that properties of Turán determinants of which entries are orthogonal polynomial families and special functions have been discussed also in the papers [21, 23, 26]. See also the book [20] for more details.

Now, observe that by using (3.12) and (4.4) we easily obtain

$$\begin{aligned} {}_1\text{Det}_n(x) &= \frac{1}{(n+1)!} \int_{[0,\infty)^{n+1}} \exp \left(-x \sum_{j=0}^n t_j \right) \\ &\quad \times \prod_{0 \leq j < k \leq n} \left(\frac{t_j}{t_j+1} - \frac{t_k}{t_k+1} \right)^2 \prod_{j=0}^n t_j^{a-1} (1+t_j)^{c-a-1} dt_j \end{aligned}$$

and

$$\begin{aligned} {}_2\text{Det}_n(x) &= \frac{1}{(n+1)!} \int_{[0,\infty)^{n+1}} \exp \left(-x \sum_{j=0}^n t_j \right) \\ &\quad \times \prod_{0 \leq j < k \leq n} (t_j - t_k)^2 \prod_{j=0}^n t_j^{a-1} (1+t_j)^{c-a-1} dt_j \end{aligned}$$

which clearly imply (4.1) and (4.2).

Furthermore, if we consider the determinants

$${}_3\text{Det}_n(x) := \begin{vmatrix} h(a, c, x) & h(a+1, c, x) & \dots & h(a+n, c, x) \\ h(a+1, c, x) & h(a+2, c, x) & \dots & h(a+n+1, c, x) \\ \vdots & \vdots & & \vdots \\ h(a+n, c, x) & h(a+n+1, c, x) & \dots & h(a+2n, c, x) \end{vmatrix}$$

and

$${}_4\text{Det}_n(x) := \begin{vmatrix} h(a, c, x) & h(a, c+1, x) & \dots & h(a, c+n, x) \\ h(a, c+1, x) & h(a, c+2, x) & \dots & h(a, c+n+1, x) \\ \vdots & \vdots & & \vdots \\ h(a, c+n, x) & h(a, c+n+1, x) & \dots & h(a, c+2n, x) \end{vmatrix},$$

where $h(a, c, x) := \Gamma(1+a-c)\psi(a, c, x)$, then by using (3.19) and (4.4) we obtain

$$\begin{aligned} {}_3\text{Det}_n(x) &= \frac{1}{(n+1)!} \int_{[0,\infty)^{n+1}} \exp \left(-x \sum_{j=0}^n t_j \right) \\ &\quad \times \prod_{0 \leq j < k \leq n} \left(\frac{t_j}{t_j+1} - \frac{t_k}{t_k+1} \right)^2 \prod_{j=0}^n x^{1-c} t_j^{a-c} (1+t_j)^{-a} dt_j \end{aligned}$$

and

$$\begin{aligned} {}_4\text{Det}_n(x) &= \frac{1}{(n+1)!} \int_{[0,\infty)^{n+1}} \exp \left(-x \sum_{j=0}^n t_j \right) \\ &\quad \times \prod_{0 \leq j < k \leq n} \left(\frac{1}{xt_j} - \frac{1}{xt_k} \right)^2 \prod_{j=0}^n x^{1-c} t_j^{a-c} (1+t_j)^{-a} dt_j. \end{aligned}$$

Similarly, if we consider the determinants

$${}_5\text{Det}_n(x) := \begin{vmatrix} g(a, c, x) & g(a+1, c+1, x) & \dots & g(a+n, c+n, x) \\ g(a+1, c+1, x) & g(a+2, c+2, x) & \dots & g(a+n+1, c+n+1, x) \\ \vdots & \vdots & & \vdots \\ g(a+n, c+n, x) & g(a+n+1, c+n+1, x) & \dots & g(a+2n, c+2n, x) \end{vmatrix}$$

and

$${}_6\text{Det}_n(x) := \begin{vmatrix} h(a, c, x) & h(a+1, c+1, x) & \dots & h(a+n, c+n, x) \\ h(a+1, c+1, x) & h(a+2, c+2, x) & \dots & h(a+n+1, c+n+1, x) \\ \vdots & \vdots & & \vdots \\ h(a+n, c+n, x) & h(a+n+1, c+n+1, x) & \dots & h(a+2n, c+2n, x) \end{vmatrix},$$

then by using (3.12), (3.19) and (4.4) we get

$$\begin{aligned} {}_5\text{Det}_n(x) &= \frac{1}{(n+1)!} \int_{[0, \infty)^{n+1}} \exp \left(-x \sum_{j=0}^n t_j \right) \\ &\quad \times \prod_{0 \leq j < k \leq n} (t_j - t_k)^2 \prod_{j=0}^n t_j^{a-1} (1+t_j)^{c-a-1} dt_j \end{aligned}$$

and

$$\begin{aligned} {}_6\text{Det}_n(x) &= \frac{1}{(n+1)!} \int_{[0, \infty)^{n+1}} \exp \left(-x \sum_{j=0}^n t_j \right) \\ &\quad \times \prod_{0 \leq j < k \leq n} \left(\frac{1}{x(t_j+1)} - \frac{1}{x(t_k+1)} \right)^2 \prod_{j=0}^n x^{1-c} t_j^{a-c} (1+t_j)^{-a} dt_j. \end{aligned}$$

Finally, if we consider the determinants

$${}_7\text{Det}_n(x) := \begin{vmatrix} g(a, c, x) & g(a+1, c+2, x) & \dots & g(a+n, c+2n, x) \\ g(a+1, c+2, x) & g(a+2, c+4, x) & \dots & g(a+n+1, c+2n+2, x) \\ \vdots & \vdots & & \vdots \\ g(a+n, c+2n, x) & g(a+n+1, c+2n+2, x) & \dots & g(a+2n, c+4n, x) \end{vmatrix}$$

and

$${}_8\text{Det}_n(x) := \begin{vmatrix} h(a, c, x) & h(a+1, c+2, x) & \dots & h(a+n, c+2n, x) \\ h(a+1, c+2, x) & h(a+2, c+4, x) & \dots & h(a+n+1, c+2n+2, x) \\ \vdots & \vdots & & \vdots \\ h(a+n, c+2n, x) & h(a+n+1, c+2n+2, x) & \dots & h(a+2n, c+4n, x) \end{vmatrix},$$

then by using again (3.12), (3.19) and (4.4) we obtain

$$\begin{aligned} {}_7\text{Det}_n(x) &= \frac{1}{(n+1)!} \int_{[0, \infty)^{n+1}} \exp \left(-x \sum_{j=0}^n t_j \right) \\ &\quad \times \prod_{0 \leq j < k \leq n} (t_j - t_k)^2 (t_j + t_k + 1)^2 \prod_{j=0}^n t_j^{a-1} (1+t_j)^{c-a-1} dt_j \end{aligned}$$

and

$$\begin{aligned} {}_8\text{Det}_n(x) &= \frac{1}{(n+1)!} \int_{[0, \infty)^{n+1}} \exp \left(-x \sum_{j=0}^n t_j \right) \\ &\quad \times \prod_{0 \leq j < k \leq n} \left(\frac{1}{xt_j(t_j+1)} - \frac{1}{xt_k(t_k+1)} \right)^2 \prod_{j=0}^n x^{1-c} t_j^{a-c} (1+t_j)^{-a} dt_j. \end{aligned}$$

Now, taking into account the well-known fact that the product of completely monotonic functions is also completely monotonic, the above integral representations imply the following result, which complements [23, Theorem 2.8].

Theorem 6. *Let $n \in \{0, 1, \dots\}$. If $a + 1 > c > 1$, then the determinants ${}_3\text{Det}_n(x)$, ${}_4\text{Det}_n(x)$, ${}_6\text{Det}_n(x)$ and ${}_8\text{Det}_n(x)$ are completely monotonic on $(0, \infty)$ with respect to x . Moreover, the determinants ${}_5\text{Det}_n(x)$ and ${}_7\text{Det}_n(x)$ are also completely monotonic on $(0, \infty)$ with respect to x for all $a > 0$ and $c \in \mathbb{R}$.*

Remark 8. We note that the above results complement the main results of section 2. To see this in what follows we will discuss the particular cases of the above results. Since for admissible values of a and c (given in Theorem 6) and for $n \in \{0, 1, \dots\}$ the functions $x \mapsto {}_i\text{Det}_n(x)$, where $i \in \{3, \dots, 8\}$, are completely monotonic on $(0, \infty)$, for those values of a and c and for all $n, m \in \{0, 1, \dots\}$ and $x > 0$ we have

$$(4.5) \quad (-1)^m {}_i\text{Det}_n^{(m)}(x) > 0.$$

We note that if $i = 3$ and we choose in (4.5) the values $m = 0$, $n = 1$ and instead of a we write $a - 1$, then we reobtain for $a > c > 1$ and $x > 0$ the left-hand side of the sharp Turán type inequality (3.15). Similarly, if $i = 4$ and we take in (4.5) the values $m = 0$, $n = 1$ and we write $c - 1$ instead of c , then for all $a + 1 > c > 2$ and $x > 0$ we get the Turán type inequality

$$\psi^2(a, c, x) - \psi(a, c - 1, x)\psi(a, c + 1, x) < \frac{1}{a - c + 1}\psi^2(a, c, x),$$

however, this is weaker than the right-hand side of the sharp inequality (3.16) or (3.17). Moreover, by using the inequality (4.5) for $i = 6$, $m = 0$, $n = 1$ and then changing a with $a - 1$ and c with $c - 1$, we obtain that the right-hand of the sharp Turán type inequality (3.4) is also valid for $a + 1 > c > 2$ and $x > 0$. Now, let $i = 8$. Then by using again the inequality (4.5) for $m = 0$, $n = 1$ and then changing a with $a - 1$ and c with $c - 2$, we obtain the Turán type inequality

$$\psi^2(a, c, x) - \psi(a - 1, c - 2, x)\psi(a + 1, c + 2, x) < \frac{1}{a - c + 1}\psi^2(a, c, x),$$

which is valid for all $a + 1 > c > 3$ and $x > 0$. This resembles to (3.3). Analogously, if $i = 5$ and we take in (4.5) the values $m = 0$, $n = 1$ and we write $a - 1$ instead of a , $c - 1$ instead of c , then for all $a > 1$, $c \in \mathbb{R}$ and $x > 0$ we get the Turán type inequality

$$\psi^2(a, c, x) - \psi(a, c - 1, x)\psi(a, c + 1, x) < \frac{1}{a}\psi^2(a, c, x),$$

however, this is weaker than the right-hand side of the sharp inequality (3.16) or (3.17). Finally, let $i = 7$. By using again the inequality (4.5) for $m = 0$, $n = 1$ and then changing a with $a - 1$ and c with $c - 2$, we obtain the Turán type inequality

$$(4.6) \quad \psi^2(a, c, x) - \psi(a - 1, c - 2, x)\psi(a + 1, c + 2, x) < \frac{1}{a}\psi^2(a, c, x),$$

which is valid for all $a > 1$, $c \in \mathbb{R}$ and $x > 0$. This also resembles to (3.3). Moreover, if we take in (4.6) instead of a the value $a + 1/2$ and instead of c the value $2a + 1$, then the above inequality becomes

$$\Delta_a(x) < \frac{1}{a + 1/2}\psi^2\left(a + \frac{1}{2}, 2a + 1, x\right),$$

which is valid for all $a > 1/2$ and $x > 0$. This complements the left-hand side of (3.3), however, it is weaker than the right-hand side of (3.3).

Remark 9. It should be mentioned here that similar results to those mentioned in Theorem 6 are also valid for the Kummer confluent hypergeometric $\Phi(a, c, \cdot)$. Namely, if we consider the determinant

$${}_9\text{Det}_n(x) := \begin{vmatrix} u(a, c, x) & u(a, c + 1, x) & \cdots & u(a, c + n, x) \\ u(a, c + 1, x) & u(a, c + 2, x) & \cdots & u(a, c + n + 1, x) \\ \vdots & \vdots & & \vdots \\ u(a, c + n, x) & u(a, c + n + 1, x) & \cdots & u(a, c + 2n, x) \end{vmatrix},$$

where

$$u(a, c, x) := \frac{\Gamma(c-a)}{\Gamma(c)} \Phi(a, c, x) = \frac{1}{\Gamma(a)} \int_0^1 e^{xt} t^{a-1} (1-t)^{c-a-1} dt,$$

then applying (4.4) we get

$$\begin{aligned} {}_9\text{Det}_n(x) &= \frac{1}{(n+1)!} \frac{1}{\Gamma^{n+1}(a)} \int_{[0,1]^{n+1}} \exp \left(x \sum_{j=0}^n t_j \right) \\ &\quad \times \prod_{0 \leq j < k \leq n} (t_k - t_j)^2 \prod_{j=0}^n t_j^{a-1} (1-t_j)^{c-a-1} dt_j. \end{aligned}$$

This in turn implies a known result of Ismail and Laforgia [23, Theorem 2.7]. Namely, for $c > a > 0$ and $n \in \{0, 1, \dots\}$ the determinant ${}_9\text{Det}_n(x)$ as a function of x is absolutely monotonic on $(0, \infty)$, i.e. for all $n, m \in \{0, 1, \dots\}$, $c > a > 0$ and $x > 0$ we have

$${}_9\text{Det}_n^{(m)}(x) \geq 0.$$

Moreover, by using again (4.4) the determinant

$${}_{10}\text{Det}_n(x) := \begin{vmatrix} v(a, c, x) & v(a+1, c, x) & \dots & v(a+n, c, x) \\ v(a+1, c, x) & v(a+2, c, x) & \dots & v(a+n+1, c, x) \\ \vdots & \vdots & \ddots & \vdots \\ v(a+n, c, x) & v(a+n+1, c, x) & \dots & v(a+2n, c, x) \end{vmatrix},$$

where $v(a, c, x) := \Gamma(a)\Gamma(c-a)\Phi(a, c, x)$, can be rewritten as

$$\begin{aligned} {}_{10}\text{Det}_n(x) &= \frac{\Gamma^{n+1}(c)}{(n+1)!} \int_{[0,1]^{n+1}} \exp \left(x \sum_{j=0}^n t_j \right) \\ &\quad \times \prod_{0 \leq j < k \leq n} \left(\frac{t_j}{1-t_j} - \frac{t_k}{1-t_k} \right)^2 \prod_{j=0}^n t_j^{a-1} (1-t_j)^{c-a-1} dt_j. \end{aligned}$$

Consequently for $c > a > 0$ and $n \in \{0, 1, \dots\}$ the determinant ${}_{10}\text{Det}_n(x)$ as a function of x is also absolutely monotonic on $(0, \infty)$, and thus for all $n, m \in \{0, 1, \dots\}$, $c > a > 0$ and $x > 0$ we have

$${}_{10}\text{Det}_n^{(m)}(x) \geq 0.$$

Now, if we take $m = 0$ and $n = 1$, and we change a to $a - 1$ we obtain the following Turán type inequality

$$\frac{\Phi(a-1, c, x)\Phi(a+1, c, x)}{\Phi^2(a, c, x)} \geq \frac{(a-1)(c-a-1)}{a(c-a)},$$

where $c > a + 1 > 2$ and $x > 0$. This inequality complements the result of Barnard et al. [15, Corollary 2], is similar to the result of Karp [27, Corollary 3] and it is weaker than the left-hand side of [28, Eq. (10)].

Finally, we also note that the determinant

$${}_{11}\text{Det}_n(x) := \begin{vmatrix} w(a, c, x) & w(a+1, c+1, x) & \dots & w(a+n, c+n, x) \\ w(a+1, c+1, x) & w(a+2, c+2, x) & \dots & w(a+n+1, c+n+1, x) \\ \vdots & \vdots & \ddots & \vdots \\ w(a+n, c+n, x) & w(a+n+1, c+n+1, x) & \dots & w(a+2n, c+2n, x) \end{vmatrix},$$

where

$$w(a, c, x) := \frac{\Gamma(a)\Gamma(c-a)}{\Gamma(a)} \Phi(a, c, x) = \int_0^1 e^{xt} t^{a-1} (1-t)^{c-a-1} dt,$$

can be rewritten as follows

$$\begin{aligned} {}_{11}\text{Det}_n(x) &= \frac{1}{(n+1)!} \int_{[0,1]^{n+1}} \exp \left(x \sum_{j=0}^n t_j \right) \\ &\quad \times \prod_{0 \leq j < k \leq n} (t_j - t_k)^2 \prod_{j=0}^n t_j^{a-1} (1-t_j)^{c-a-1} dt_j. \end{aligned}$$

This in turn implies that for $c > a > 0$ and $n \in \{0, 1, \dots\}$ the determinant ${}_{11}\text{Det}_n(x)$ as a function of x is also absolutely monotonic on $(0, \infty)$, and thus for all $n, m \in \{0, 1, \dots\}$, $c > a > 0$ and $x > 0$ we have

$${}_{11}\text{Det}_n^{(m)}(x) \geq 0.$$

Now, if we take in this inequality $m = 0$ and $n = 1$, and we change a to $a - 1$, c to $c - 1$ we obtain the following Turán type inequality

$$\frac{\Phi(a-1, c-1, x) \Phi(a+1, c+1, x)}{\Phi^2(a, c, x)} \geq \frac{(a-1)c}{(c-1)a},$$

where $c > a > 1$ and $x > 0$. This is a particular case of the left-hand side inequality of [27, Corollary 3] and is the counterpart of [27, Theorem 1]. See also [7, Theorem 2] for a similar Turán type inequality.

We end the paper with the following remark.

Remark 10. We note that many other special functions have representations of the type (4.3). For example, the modified Bessel functions of the first and second kind I_a and K_a are like this. Thus, by using the idea used in this section we can explore this further to get positivity of many determinants which entries are special functions. Also we may be able to prove that the determinants are completely (or absolutely) monotonic if the integral representation (4.3) involves completely (absolutely) monotonic kernels. See [13] for similar results on the so-called Krätzel function.

Now, we would like to complement the results of Theorem 2.3 and 2.5 from [23]. For this we consider first for $a > -1/2$ and $x > 0$ the integral representation (see [1, p. 376] or [41, p. 172])

$$K_a(x) = \frac{\sqrt{\pi} \left(\frac{x}{2}\right)^a}{\Gamma\left(a + \frac{1}{2}\right)} \int_1^\infty e^{-xt} (t^2 - 1)^{a-\frac{1}{2}} dt$$

and apply Theorem 5 for

$$f_n(x) = \frac{\Gamma\left(a + n + \frac{1}{2}\right)}{\sqrt{\pi} \left(\frac{x}{2}\right)^{a+n}} e^x K_{a+n}(x)$$

to get

$$\begin{aligned} {}_{12}\text{Det}_n(x) &= \begin{vmatrix} u_a(x) & u_{a+1}(x) & \cdots & u_{a+n}(x) \\ u_{a+1}(x) & u_{a+2}(x) & \cdots & u_{a+n+1}(x) \\ \vdots & \vdots & & \vdots \\ u_{a+n}(x) & u_{a+n+1}(x) & \cdots & u_{a+2n}(x) \end{vmatrix} \\ &= \frac{1}{(n+1)!} \int_{[1,\infty)^{n+1}} \exp \left(-x \sum_{j=0}^n (t_j - 1) \right) \prod_{0 \leq j < k \leq n} (t_j^2 - t_k^2)^2 \prod_{j=0}^n (t_j^2 - 1)^{a-\frac{1}{2}} dt_j, \end{aligned}$$

where

$$u_a(x) := \frac{\Gamma\left(a + \frac{1}{2}\right)}{\sqrt{\pi} \left(\frac{x}{2}\right)^a} e^x K_a(x).$$

Clearly, the determinant ${}_{12}\text{Det}_n(x)$ is completely monotonic on $(0, \infty)$ for each $n \in \{0, 1, \dots\}$ and $a > -1/2$. Now, for $a > -1/2$ consider the integral representation (see [1, p. 376] or [41, p. 79])

$$I_a(x) = \frac{\left(\frac{x}{2}\right)^a}{\sqrt{\pi}\Gamma\left(a + \frac{1}{2}\right)} \int_{-1}^1 e^{\pm xt} (1 - t^2)^{a-\frac{1}{2}} dt.$$

Here we have two choices for f_n . They are

$$f_n(x) = \frac{\sqrt{\pi}\Gamma\left(a + n + \frac{1}{2}\right)}{\left(\frac{x}{2}\right)^{a+n}} e^x I_{a+n}(x)$$

or

$$f_n(x) = \frac{\sqrt{\pi}\Gamma\left(a + n + \frac{1}{2}\right)}{\left(\frac{x}{2}\right)^{a+n}} e^{-x} I_{a+n}(x).$$

One is led to an absolutely monotonic determinant and one to completely monotonic determinant. More precisely, as a function of x and for $a > -1/2$ and $n \in \{0, 1, \dots\}$ the determinant

$$\begin{aligned} {}_{13}\text{Det}_n(x) &= \begin{vmatrix} v_a(x) & v_{a+1}(x) & \cdots & v_{a+n}(x) \\ v_{a+1}(x) & v_{a+2}(x) & \cdots & v_{a+n+1}(x) \\ \vdots & \vdots & & \vdots \\ v_{a+n}(x) & v_{a+n+1}(x) & \cdots & v_{a+2n}(x) \end{vmatrix} \\ &= \frac{1}{(n+1)!} \int_{[-1,1]^{n+1}} \exp\left(x \sum_{j=0}^n (1-t_j)\right) \prod_{0 \leq j < k \leq n} (t_j^2 - t_k^2)^2 \prod_{j=0}^n (t_j^2 - 1)^{a-\frac{1}{2}} dt_j, \end{aligned}$$

where

$$v_a(x) := \frac{\sqrt{\pi}\Gamma\left(a + \frac{1}{2}\right)}{\left(\frac{x}{2}\right)^a} e^x I_a(x) = \int_{-1}^1 e^{(1-t)x} (1 - t^2)^{a-\frac{1}{2}} dt,$$

is absolutely monotonic on $(0, \infty)$, while the determinant

$$\begin{aligned} {}_{14}\text{Det}_n(x) &= \begin{vmatrix} w_a(x) & w_{a+1}(x) & \cdots & w_{a+n}(x) \\ w_{a+1}(x) & w_{a+2}(x) & \cdots & w_{a+n+1}(x) \\ \vdots & \vdots & & \vdots \\ w_{a+n}(x) & w_{a+n+1}(x) & \cdots & w_{a+2n}(x) \end{vmatrix} \\ &= \frac{1}{(n+1)!} \int_{[-1,1]^{n+1}} \exp\left(-x \sum_{j=0}^n (1-t_j)\right) \prod_{0 \leq j < k \leq n} (t_j^2 - t_k^2)^2 \prod_{j=0}^n (t_j^2 - 1)^{a-\frac{1}{2}} dt_j, \end{aligned}$$

where

$$w_a(x) := \frac{\sqrt{\pi}\Gamma\left(a + \frac{1}{2}\right)}{\left(\frac{x}{2}\right)^a} e^{-x} I_a(x) = \int_{-1}^1 e^{-(1-t)x} (1 - t^2)^{a-\frac{1}{2}} dt,$$

is completely monotonic on $(0, \infty)$. We mention here that for $n = 1$ the above results lead to weak Turán type inequalities for modified Bessel functions of the first and second kinds, which were mentioned already in Remarks 2.4 and 2.6 in [23].

Finally, it is worth to mention that the above method can be used also to prove absolute and complete monotonic properties of determinants whose entries are probability density functions. For example, for the probability density function of the non-central chi distribution we can prove such results. More precisely, if we consider the probability density function $\chi_{a,\tau} : (0, \infty) \rightarrow (0, \infty)$ of the non-central chi distribution (see [25]) with shape parameter $a > 0$ and non-centrality parameter $\tau > 0$, defined by

$$\chi_{a,\tau}(x) := \tau e^{-\frac{x^2+\tau^2}{2}} \left(\frac{x}{\tau}\right)^{\frac{a}{2}} I_{\frac{a}{2}-1}(\tau x),$$

then it is not difficult to see that as a function of x and for all $a > 1/2$, $\tau > 0$ and $n \in \{0, 1, \dots\}$ the determinant

$$\begin{aligned} {}_{15}\text{Det}_n(x) &= \begin{vmatrix} r_{2a}(x) & r_{2a+1}(x) & \cdots & r_{2a+n}(x) \\ r_{2a+1}(x) & r_{2a+2}(x) & \cdots & r_{2a+n+1}(x) \\ \vdots & \vdots & & \vdots \\ r_{2a+n}(x) & r_{2a+n+1}(x) & \cdots & r_{2a+2n}(x) \end{vmatrix} \\ &= \frac{1}{(n+1)!} \int_{[-1,1]^{n+1}} \exp\left(\tau x \sum_{j=0}^n (1-t_j)\right) \prod_{0 \leq j < k \leq n} (t_j^2 - t_k^2)^2 \prod_{j=0}^n (t_j^2 - 1)^{a-\frac{3}{2}} dt_j, \end{aligned}$$

where

$$r_a(x) := \frac{\sqrt{\pi} 2^{\frac{a}{2}-1} \Gamma\left(\frac{a-1}{2}\right)}{x^{a-1}} e^{\frac{(x+\tau)^2}{2}} \chi_{a,\tau}(x) = \int_{-1}^1 e^{(1-t)\tau x} (1-t^2)^{\frac{a-3}{2}} dt,$$

is absolutely monotonic on $(0, \infty)$, while the determinant

$$\begin{aligned} {}_{16}\text{Det}_n(x) &= \begin{vmatrix} s_{2a}(x) & s_{2a+1}(x) & \cdots & s_{2a+n}(x) \\ s_{2a+1}(x) & s_{2a+2}(x) & \cdots & s_{2a+n+1}(x) \\ \vdots & \vdots & & \vdots \\ s_{2a+n}(x) & s_{2a+n+1}(x) & \cdots & s_{2a+2n}(x) \end{vmatrix} \\ &= \frac{1}{(n+1)!} \int_{[-1,1]^{n+1}} \exp\left(-\tau x \sum_{j=0}^n (1-t_j)\right) \prod_{0 \leq j < k \leq n} (t_j^2 - t_k^2)^2 \prod_{j=0}^n (t_j^2 - 1)^{a-\frac{3}{2}} dt_j, \end{aligned}$$

where

$$s_a(x) := \frac{\sqrt{\pi} 2^{\frac{a}{2}-1} \Gamma\left(\frac{a-1}{2}\right)}{x^{a-1}} e^{\frac{(x-\tau)^2}{2}} \chi_{a,\tau}(x) = \int_{-1}^1 e^{-(1-t)\tau x} (1-t^2)^{\frac{a-3}{2}} dt,$$

is completely monotonic on $(0, \infty)$. Note that the positivity of the above determinants for $n = 1$ yields the Turán type inequality

$$\chi_{2a+1,\tau}^2(x) - \chi_{2a,\tau}(x)\chi_{2a+2,\tau}(x) < \left[1 - \frac{\Gamma^2(a)}{\Gamma\left(a - \frac{1}{2}\right)\Gamma\left(a + \frac{1}{2}\right)}\right] \chi_{2a+1,\tau}^2(x),$$

where $a > 1/2$, $\tau > 0$ and $x > 0$. This inequality completes [5, Theorem 2.3], however it is weaker than the sharp Turán type inequality

$$0 < \chi_{2a+1,\tau}^2(x) - \chi_{2a,\tau}(x)\chi_{2a+2,\tau}(x) < \left[1 - \frac{\Gamma^2\left(a + \frac{1}{2}\right)}{\Gamma(a)\Gamma(a+1)}\right] \chi_{2a+1,\tau}^2(x),$$

which holds for all $a, \tau, x > 0$ and is a particular case of [11, Theorem 2.4].

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